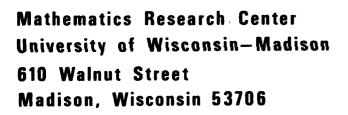MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

A MULTIPLICITY RESULT FOR A
SEMILINEAR DIRICHLET PROBLEM

J. V. A. Goncalves

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

April 1983

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

A MULTIPLICITY RESULT FOR A SEMILINEAR DIRICHLET PROBLEM

J. V. A. Goncalves*

Technical Summary Report #2515
April 1983

ABSTRACT

Let Ω be a bounded smooth domain in R^N. We give an estimate for the number of solutions of the problem

$$Lu + g(u) = \lambda u \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0$$

where L is a second order elliptic operator. The behavior of the nonlinearity g both at 0 and at ∞ and the relationship between λ and the spectrum of L play an important role in the analysis.

* Universidade de Brasilia. Partially supported by CNPq/Brasil.

SIGNIFICANCE AND EXPLANATION

In this paper we consider the number of solutions of the Dirichlet problem for semilinear elliptic equations. Specifically we study the question of finding solutions u of an equation such as $-\Delta u + g(u) = \lambda u$ in a bounded domain $\Omega \subset \mathbb{R}^N$ subject to the condition that u vanishes on the boundary of Ω. This problem has been intensively studied in the last few years; it arises in many situations such as nonlinear diffusion generated by nonlinear sources, the thermal ignition of gases, and others. In this paper we derive precise estimates of the number of solutions under assumptions which are natural for these problems thereby complementing results obtained by a number of authors.

A MULTIPLICITY RESULT FOR A SEMILINEAR DIRICHLET PROBLEM

J. V. A. Goncalves[*]

1. Introduction

Let Ω be a bounded smooth domain in R^N. We are concerned with the question of existence of multiple solutions for the problem

$$(1) \qquad Lu + g(u) = \lambda u \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0$$

where $g: R \to R$ is a locally Lipschitz continuous function, λ is a real parameter and

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) + a_0 u$$

has smooth real valued coefficient functions $a_{ij} = a_{ji}$ and $a_0 > 0$ satisfying the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq c \sum_{i=1}^{N} \xi_i^2 \quad x \in \Omega, \ (\xi_1,\dots,\xi_N) \in R^N$$

for some postive constant c. Let $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_j \leq \dots$ be the sequence of eigenvalues of the linear problem

$$Lu = \lambda u \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0$$

with each λ_j occurring in the sequence as often as its multiplicity. We recall that the corresponding sequence of eigenfunctions $\varphi_1, \varphi_2, \dots, \varphi_j, \dots$ is a complete orthonormal system of $L^2(\Omega)$ with $\varphi_1 > 0$ in Ω and $\frac{\partial \varphi_1}{\partial n} < 0$

on $\partial\Omega$ where $n(x)$ is the outward pointing normal to $\partial\Omega$. Our goal in this paper is to prove the following result.

Theorem 1. **Assume** $g: R \to R$ **is a locally Lipschitz continuous function satisfying**

(g_1) $\qquad\qquad\qquad\qquad g(z) = o(|z|)$ **at** $z = 0$

(g_2) $\qquad\qquad\qquad\qquad \dfrac{g(z)}{z} \to +\infty$ **as** $|z| \to \infty$

and

(g_3) $\qquad\qquad\qquad 0 < G(z) \equiv \int_0^z g(t)dt < \dfrac{1}{2} zg(z), \quad z \in R.$

Then (1) **has at least four solutions provided** $\lambda > \lambda_2$.

Theorem 1 is an improvement to an earlier result by Struwe [5] where a condition stronger than (g_3), namely

(g_3') $\qquad \dfrac{g(z)}{z}$ is increasing in $(0, +\infty)$ and is decreasing in $(-\infty, 0)$

is assumed. As a matter of fact, by requiring more regularity on g, namely $g \in C^1(R)$, Ambrosetti [6] proved as an application of the Morse Theory that at least 3 nontrivial solutions of (1) do exist provided $\lambda > \lambda_2$ and $(g_1) - (g_2)$ hold. Also for $g \in C^1(R)$ it was shown by Hofer [7] that (1) has at least 4 nontrivial solutions if $(g_1) - (g_2)$ hold and $\lambda \in (\lambda_i, \lambda_{i+1})$, $i > 2$. We recall that more precise results are known both in the O.D.E. case (see Berestycki [11] and references therein) and in the case that g is odd (see Rabinowitz [4] where the technique used to prove Th. 2.8 applies to (1)). We refer the reader to [1], [3], [9], [10] for additional results concerning (1).

Remarks.

(i) Theorem 1 holds if (g_2) is replaced by the weaker condition

 (g_2') there exist numbers $z_- < 0 < z_+$ such that

$$\lambda z_- - g(z_-) = 0 = \lambda z_+ - g(z_+).$$

(ii) If we assume (g_3') in Th. 1 then it can be shown that (1) has exactly a positive solution and a negative one. Moreover (1) has a maximal solution and a minimal solution (see e.g. the Lecture Notes by deFigueiredo [10]).

(iii) Dependence of g on $x \in \Omega$ is allowed by modifying conveniently our assumptions, for example, by requiring the limits $(g_1) - (g_2)$ to be uniform with respect to $x \in \Omega$.

(iv) If $zg(z) > 0$ for $z \neq 0$, then it follows easily that (1) admits only the trivial solution for $\lambda < \lambda_1$.

2. Notations and Preliminary Results

We introduce in the Sobolev space $H = H_0^1(\Omega)$ the inner product

$$(u,v)_1 = \sum_{i,j=1}^{N} \int_\Omega \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv \right) dx$$

$$= B(u,v) + \int_\Omega a_0 uv\, dx \quad u, v \in H.$$

Let $|u|_1^2 = (u, u)_1$ for $u \in H$ be the corresponding norm which is equivalent to the usual norm of H. We denote by $(\cdot,\cdot)_0$ the inner product in $L^2(\Omega)$ and by $|\cdot|_0$ the corresponding norm and recall that $(v, v)_1 \leq \lambda_j (v,v)_0$ for $v \in V_j$ where $V_j = \text{span}\{\varphi_1,\ldots,\varphi_j\}$ and $(w,w)_1 \geq \lambda_{j+1}(w,w)_0$ for $w \in W_j$ where $W_j = \text{span}\{\varphi_{j+1},\ldots\}$. Moreover $H = V_j \oplus W_j$ and V_j is orthogonal to W_j both with respect to the L^2 and H inner products. We denote by P_j and Q_j the projectors onto V_j and W_j respectively.

Next we will associate to (1) an auxiliary problem. Let

$$z_+ = \sup\{z > 0;\ \lambda t - g(t) > 0,\ 0 < t < z\}$$

and

$$z_- = \inf\{z < 0;\ \lambda t - g(t) < 0,\ z < t < 0\}.$$

It follows from $(g_1) - (g_2)$ that $-\infty < z_- < 0 < z_+ < +\infty$. Instead of working directly with (1) we are going to look for solutions of

$$(1') \qquad\qquad Lu = f(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

where $f: \mathbb{R} \to \mathbb{R}$ is the bounded function (f is Lipschitz continuous if g is locally Lipschitz continuous) defined by $f(z) = \lambda z - g(z)$ if $z_- < z < z_+$ and $f(z) = 0$ otherwise.

-4-

Lemma 1. _Let_ $g: R \to R$ _be a continuous function satisfying_ (g_2). _If_ $u \in H$ _is a solution of_ (1') _then_ $z_- < u < z_+$ _in_ Ω _and_ u _is a solution of_ (1).

Proof of Lemma 1. Let $u_{z_+}(x) = u(x)$ if $u(x) < z_+$ and $u_{z_+}(x) = z_+$ if $u(x) > z_+$. By a theorem of Stampacchia [12], $u_{z_+} \in H$. On the other hand, $(u, u_{z_+} - u)_1 = \int_\Omega f(u)(u - u_{z_+}) < 0$, $B(u_{z_+}, u - u_{z_+}) = 0$ and $\int_\Omega a_0(u - u_{z_+}) > 0$. By using the uniform ellipticity of L and Poincaré's Inequality we get $c|u_{z_+} - u|_0^2 < 0$ for some positive constant c. So $u < z_+$ in Ω. Similarly we can show that $z_- < u$ in Ω. Lemma 1 is proved.

Let $F(z) = \int_0^z f(t)dt$ for $z \in R$ and

$$J(u) = \frac{1}{2}|u|_1^2 - \int_\Omega F(u), \quad u \in H.$$

It follows that $J \in C^1(H,R)$, $(\nabla J(u), v)_1 = (u,v)_1 - \int_\Omega f(u)v \quad u, v \in H$ and actually the solutions of (1') are the critical points of J. The following auxiliary result will play an important role in this paper.

Theorem 2. _Let_ $g: R \to R$ _be a continuous function satisfying_ $(g_1) - (g_2)$. _Suppose_ $\lambda > \lambda_1$. _Then there exist solutions_ $u_\pm \in C^{1+\alpha}(\bar{\Omega})$, $(\alpha \in (0,1))$, _of_ '1') _with_ $u_- < 0 < u_+$ _in_ Ω, $\frac{\partial u_+}{\partial n} < 0$ _on_ $\partial\Omega$ _and_ $\frac{\partial u_-}{\partial n} > 0$ _on_ $\partial\Omega$. _Moreover_ $J(u_+) < J(u)$ _if_ $u > 0$ _and_ $J(u_-) < J(u)$ _if_ $u < 0$.

Proof of Theorem 2. Let $f_+(z) = \lambda z - g(z)$ if $0 < z < z_+$ and $f_+(z) = 0$ otherwise. Let $F_+(z) = \int_0^z f_+(t)dt$ for $z \in R$ and

$$J_+(u) = \frac{1}{2}|u|_1^2 - \int_\Omega F_+(u), \quad u \in H.$$

It is quite easy to check that $J_+ \in C^1(H,R)$, $J_+(u) \to +\infty$ as $|u|_1 \to \infty$ and J_+ satisfies the Palais-Smale condition ((PS) for short). Thus there exists $u_+ \in H$ such that $J_+(u_+) = \inf_{u \in H} J_+(u)$. Actually, by using (g_1) and $\lambda > \lambda_1$

we find that $J(t\varphi_1) < 0$ for all $t > 0$ near 0. On the other hand, by recalling that $Lu_+ = f_+(u_+)$ in Ω in the weak sense and using the elliptic estimates, Sobolev embedding theorem and maximum principles we get $u_+ \in C^{1+\alpha}(\bar{\Omega})$ for $\alpha \in (0,1)$, $u_+ > 0$ in Ω and $\dfrac{\partial u_+}{\partial n} < 0$ on $\partial\Omega$. Moreover, $J(u_+) = J_+(u_+) < J_+(u) = J(u)$ if $u > 0$. Similar arguments apply with respect to u_-. This proves Theorem 2.

Our approach to get the third nontrivial solution consists in applying the idea of the Mountain Pass Theorem (*cf.* Ambrosetti-Rabinowitz [2]). However the fact that $u_\pm$ are not necessarily minima of J poses a technical difficulty. In order to get around it we had to use a reduction argument by Lazer-Landesman-Meyers [13] related to the global Ljapunov-Schmidt method. Actually we will use an improvement of that reduction argument by Castro [14]. Since (under the assumptions in Theorem 1) $f(z)$ is Lipschitz continuous we take $k > 2$ such that $\lambda_k < \lambda_{k+1}$ and $|f(z)-f(z')| < \lambda_k|z-z'|$ for all $z, z' \in R$. Now, if $v \in V = V_k$ and $w_1, w_2, \in W = W_k$ we get

$$\left(\nabla J(v+w_1) - \nabla J(v+w_2), w_1-w_2\right)_1 =$$

$$|w_1 - w_2|_1^2 - \int_\Omega \left(f(v+w_1) - f(v+w_2)\right)(w_1-w_2)$$

$$> \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) |w_1-w_2|_1^2$$

By Castro's result mentioned earlier ([14], Lemma 2.1), there exists $\theta \in C^0(V,W)$ satisfying

(i)   If $v \in V$ then $J(v+\theta(v)) = \inf_{w\in W} J(v+w)$ and actually, $\theta(v)$ is the only element of W such that $\left(\nabla J(v+\theta(v)), w\right)_1 = 0$ for all $w \in W$.

(ii) Let $\hat{J}(v) = J(v+\theta(v))$ for $v \in V$. Then $\hat{J} \in C^1(V, R)$ and $\left(\nabla\hat{J}(v), v_1\right)_1 = \left(\nabla J(v+\theta(v)), v_1\right)_1$ for $v, v_1 \in V$.

Moreover, $u \in H$ is a critical point of J iff $u = v + \theta(v)$ and v is critical point of $\hat{J}$. So $u_{\pm} = v_{\pm} + \theta(v_{\pm})$ where $v_{\pm}$ are critical points of $\hat{J}$.

Lemma 2. **Assume** $g: R \to R$ **is a locally Lipschitz continuous function satisfying** $(g_1) - (g_2)$. **Suppose** $\lambda > \lambda_1$. **Then there exists** $\delta_1 > 0$ **such that** $B_{\delta_1}(v_+) \cap B_{\delta_1}(v_-) = \phi$, $\hat{J}(v_+) < \hat{J}(v)$ **if** $v \in B_{\delta_1}(v_+)$ **and** $\hat{J}(v_-) < \hat{J}(v)$ **if** $v \in B_{\delta_1}(v_-)$.

Proof of Lemma 2. From (i)-(ii) above we get for all $v \in V$, $w \in W$, $u \in H$

$$\left(\theta(v), w\right)_1 = \int_{\Omega} f(v+\theta(v))w.$$

By recalling that v and w are orthogonal both with respect to L^2 and H and writing $P = P_k$ and $Q = Q_k$ we find

$$\left(\theta(v), u\right)_1 = \int_{\Omega} Qf(v+\theta(v))u$$

that is

$$L\theta(v) = Qf(v+\theta(v)) \quad \text{in} \quad \Omega$$

in the weak sense. It follows from the elliptic estimates that

$$|\theta(v) - \theta(v_+)|_{W^{2,p}} \leq c|Qf(v+\theta(v)) - Qf(v_+ + \theta(v_+))|_{L^p}$$

for any $p > 1$. On the other hand, it is quite easy to check that

$$|Qf(v + \theta(v)) - Qf(v_+ + \theta(v_+))|_{L^p} \to 0 \quad \text{as} \quad |v-v_+|_1 \to 0$$

Thus we get by using the Sobolev embedding $W^{2,p} \subseteq C^1(\bar{\Omega})$, ($p > 1$ large enough), that

$$|\theta(v) - \theta(v_+)|_{C^1(\bar{\Omega})} \to 0 \quad \text{as} \quad |v-v_+|_1 \to 0$$

and further

$$|\left(v + \theta(v)\right) - \left(v_+ + \theta(v_+)\right)|_{C^1(\bar{\Omega})} \to 0 \quad \text{as} \quad |v-v_+|_1 \to 0.$$

Therefore by recalling that $u_+ > 0$ in Ω and $\frac{\partial u_+}{\partial n} < 0$ on $\partial\Omega$ we find

that $v + \theta(v) > 0$ in Ω provided $|v-v_+|_1 < \delta_1$ with $\delta_1 > 0$ small

enough. In particular, $\widetilde{J}(v_+) = J\left(v_+ + \theta(v_+)\right) < J\left(v + \theta(v)\right) = \widetilde{J}(v)$ for all

$v \in B_{\delta_1}(v_+)$. Similar arguments apply with respect to v_-. This proves

Lemma 2.

3. Proof of Theorem 1.

We can assume that both v_- and v_+ are isolated critical points of $\widehat{J}$. Let, as in Ambrosetti-Rabinowitz [2],

$$B = \{\beta \in C^0([0,1],V);\ \beta(0) = v_-,\ \beta(1) = v_+\}$$

and

$$c = \inf_{\beta \in B}\ \sup_{0 < t < 1}\ \widehat{J}(\beta(t)).$$

In order to show that $c \in \mathbb{R}$ is a critical value of $\widehat{J}$ we recall that $\widehat{J}(v) \to +\infty$ as $|v|_1 \to \infty$. So $\widehat{J}$ satisfies the (PS) condition. Suppose by contradiction that c is not a critical value of $\widehat{J}$. It follows from the Deformation Theorem (see e.g. Clark [15], Rabinowitz [8]) that there exist $\varepsilon > 0$ and $\sigma \in C^0([0,1] \times V, V)$ such that for all $v \in V$

(i) $\sigma(0,v) = v$.

(ii) $\widehat{J}(\sigma(t,v))$ is nonincreasing in $t \in [0,1]$.

(iii) if $\widehat{J}(v) < c + \varepsilon$ then $\widehat{J}(\sigma(1,v)) < c-\varepsilon$.

On the other hand, from the definition of c there exists $\beta \in B$ such that $\widehat{J}(\beta(t)) < c + \varepsilon$ for $t \in [0,1]$. Let $\widehat{\beta}(\cdot) = \sigma(1,\beta(\cdot))$. We claim that $\widehat{\beta} \in B$. Indeed, $\widehat{\beta} \in C^0([0,1],V)$ and $\widehat{\beta}(0) = \sigma(1,v_-)$. Further there exists $\mu \in (0,1)$ such that $|\sigma(t,v_-) - \sigma(0,v_-)| < \delta_1$ for $t \in [0,\mu]$. On the other hand, $\widehat{J}(\sigma(t,v_-)) < \widehat{J}(\sigma(0,v_-)) = \widehat{J}(v_-)$ for $t \in [0,1]$. Thus $\sigma(t,v_-) = v_-$ for $t \in [0,\mu]$. If

$$\mu^* = \sup\{\mu > 0;\ \sigma(t,v_-) = v_-,\quad 0 < t < \mu\}$$

then we find easily by repeating the above argument that $\mu^* = 1$. So $\sigma(1,v_-) = v_-$ and $\widehat{\beta}(1) = v_-$. Similarly we get $\widehat{\beta}(1) = v_+$ and consequently

$\hat{\beta} \in B$. But this is impossible, since by (iii) above $\widehat{J}(\hat{\beta}(t)) < c - \varepsilon$ for $t \in [0,1]$.

Now,

$$\min_{v \in \partial B_{\delta_1}(v_+)} \widehat{J}(v) > \widehat{J}(v_+)$$

since we are assuming v_+ to be an isolated minimum of $\widehat{J}$. Therefore

$$\inf_{\beta \in B} \widehat{J}\big(\beta([0,1]) \cap \partial B_{\delta_1}(v_+)\big) > \widehat{J}(v_+)$$

and $c > \widehat{J}(v_+)$. Similarly we get $c > \widehat{J}(v_-)$. Next we will show that $c < 0$. We claim that

(*) there exists $\alpha \in C^0([0,1], H)$ such that $\alpha(0) = u_-$, $\alpha(1) = u_+$,
$u_- < \alpha(t) < u_+$ and $J(\alpha(t)) < 0$, $0 < t < 1$.

Let $\beta(t) = P \alpha(t)$ for $t \in [0,1]$, where α is given in (*). Then for all $t \in [0,1]$

$$\widehat{J}\big(\beta(t)\big) = J\big(\beta(t) + \theta(\beta(t))\big)$$

$$= \inf_{w \in W} J\big(\beta(t) + w\big)$$

$$< J\big(\beta(t) + Q \alpha(t)\big)$$

$$= J\big(\alpha(t)\big) < 0$$

so that $c < 0$. Next we will prove (*). If $t \in (0,1]$ then we get by using (g_3)

$$J(tu_-) = \frac{t^2}{2} |u_-|_1^2 - \int_\Omega F(tu_-)$$

$$= \frac{t^2}{2} |u_-|_1^2 - \frac{\lambda}{2} t^2 |u_-|_0^2 + \int_\Omega G(tu_-)$$

$$= t^2 \left(\frac{|u_-|_1^2}{2} - \frac{\lambda}{2} |u_-|_0^2 + \int_\Omega \frac{G(tu_-)}{t^2} \right)$$

$$< t^2 J(u_-) < 0.$$

Similarly, $J(tu_+) < 0$. Now, the following inequalities are consequences of the maximum principles

(iv) there exist $b_- < 0 < b_+$ such that

$$u_- < b_-\varphi_1 < 0 < b_+\varphi_1 < u_+ \quad \text{in} \quad \Omega$$

and

(v) if $a < 0 < b$ then there exists $r > 0$ such that

$$a\varphi_1 < u < b\varphi_1 \quad \text{in} \quad \Omega$$

for all $v \in \partial B_r(0) \cap V_s$ where $V_s = \text{span} \{\varphi_1, \ldots, \varphi_s\}$ and $\varphi_2, \ldots, \varphi_s$ are the eigenfunctions corresponding to λ_2.

On the other hand,

$$J(b_-\varphi_1) < 0 \quad \text{and} \quad J(-b_-u_-) < 0$$

provided $b_- < 0$ is near enough to 0. Let

$$\alpha_1^-(t) = -tb_-u_- + (1-t)b_-\varphi_1, \quad 0 < t < 1.$$

We will show that $J(\alpha_1^-(t)) < 0$ provided $b_- < 0$ is taken sufficiently close to 0. Indeed, if $t \in (0,1]$ we get

$$
\begin{aligned}
J(\alpha_1^-(t)) &= \frac{1}{2}|\alpha_1^-(t)|_1^2 - \int_\Omega F(\alpha_1^-(t)) \\
&= \frac{1}{2}|\alpha_1^-(t)|_1^2 - \frac{\lambda}{2}|\alpha_1^-(t)|_0^2 + \int_\Omega G(\alpha_1^-(t)).
\end{aligned}
$$

(2)

Now, there exists $\xi: \Omega \to \mathbb{R}$ such that

$$\alpha_1^-(t) < \xi < -tb_-u_- \quad \text{in} \quad \Omega$$

and

$$G(\alpha_1^-(t)) = G(-tb_-u_-) + (1-t)g(\xi)b_-\varphi_1.$$

Moreover,

$$b_-(\varphi_1 - u_-) < \xi < 0 \quad \text{in} \quad \Omega$$

so that $\xi \to 0$ uniformly in Ω as $b_- \to 0$. We get from (2)

$$J\left(\alpha_1^-(t)\right) < (tb_-)^2 J(u_-) + t(1-t)b_-^2 \int_\Omega g(u_-)\varphi_1 + (1-t)^2 b_-^2 \frac{\lambda_1 - \lambda}{2} + (1-t) \int_\Omega g(\xi)b_- \varphi_1$$

$$< (tb_-)^2 J(u_-) + (1-t)^2 b_-^2 \frac{\lambda_1 - \lambda}{2} + \int_\Omega g(\xi)b_- \varphi_1 .$$

Let

$$M = \sup_{0 < t < 1} \left(t^2 J(u_-) + (1-t)^2 \frac{\lambda_1 - \lambda}{2} \right).$$

It follows that $M < 0$. On the other hand,

$$\left| \int_\Omega \frac{g(\xi)}{\xi} \frac{\xi}{b_-} \varphi_1 \right| < |\varphi_1 - u_-|_\infty |\varphi_1|_\infty \int_\Omega \frac{g(\xi)}{\xi}$$

and by using (g_1) we get

$$M + \int_\Omega \frac{g(\xi)}{b_-} \varphi_1 < 0$$

provided $b_- < 0$ is sufficiently close to 0. Therefore $J\left(\alpha_1^-(t)\right) < 0$ for all $t \in [0,1]$ if $b_- < 0$ is near enough to 0. Let

$$\alpha_0^-(t) = tu_-, \quad -b_- < t < 1$$

Then $J\left(\alpha_0^-(t)\right) < 0$ for $t \in [-b_-,1]$. Similarly, we introduce

$$\alpha_1^+(t) = tb_+ u_+ + (1-t)b_+\varphi_1, \quad 0 < t < 1$$

and

$$\alpha_0^+(t) = tb_+, \quad b_+ < t < 1$$

Let $r > 0$ be as in (ii). If $r > 0$ is small enough we find that $J(v) < 0$ for $v \in \partial B_r(0) \cap V_s$ and $\partial B_r(0) \cap \text{span } \{\varphi_1\} = \{b'_-\varphi_1, b'_+\varphi_1\}$ where $0 < b'_+ < b_+$ and $b_- < b'_- < 0$. Now, take any path say α_0 on $\partial B_r(0) \cap V_s$ connecting $b'_-\varphi_1$ to $b'_+\varphi_1$. Let $\alpha_2^-(t)$ and $\alpha_2^+(t)$ be the segments

-12-

connecting $b_-\varphi_1$ to $b'_-\varphi_1$ and $b'_+\varphi_1$ to $b_+\varphi_1$. By joining all of these

paths we get after reparametrization the path α as asserted in (*). This

proves Theorem 1.

REFERENCES

[1] Ambrosetti, A. & Mancini, G. <u>Sharp nonuniqueness results for some nonlinear problems.</u> Nonlinear Analysis - T. M. A. 3 (1979) 635-645.

[2] Ambrosetti, A. & Rabinowitz, P. <u>Dual variational methods in critical point theory and applications.</u> J. Funct. Anal. 14 (1973) 349-381.

[3] Berger, M. <u>Nonlinear problems with exactly three solutions.</u> Ind. Univ. Math. J. 28 (1979) 689-698.

[4] Rabinowitz, P. <u>A minimax principle and applications to elliptic partial differential equations.</u> Lecture Notes in Math. no. 648 Springer-Verlag (1978) 97-115.

[5] Struwe, M. <u>A note on a result of Ambrosetti and Mancini</u> (preprint).

[6] Ambrosetti, A. <u>On a class of nonlinear Dirichlet problems with multiple solutions.</u> (preprint).

[7] Hofer, H. <u>Variational and topological methods in partially ordered Hilbert spaces.</u> Math. Annalen 261 (1982) 493-514.

[8] Rabinowitz, P. <u>Variational methods for nonlinear eigenvalue problems.</u> Edizione Cremonese, Rome (1974) 141-195.

[9] Lazer, A. & McKenna P. <u>On multiple solutions of a nonlinear Dirichlet problem.</u> Differential Equations-Academic Press (1980) 199-214.

[10] DeFigueiredo, D. <u>Positive solutions of semilinear elliptic problems.</u> Lecture Notes in Math. no. 957 Springer-Verlag (1982) 34-87.

[11] Berestycki, H. <u>Le nombre de solutions de certains problèmes semilinéares elliptiques.</u> J. Funct. Anal. 40 (1981) 1-29.

[12] Stampacchia, G. <u>Equations elliptiques du second ordre à coefficients discontinues.</u> Le presses de l'Université de Montréal (1966).

[13] Lazer, A.; Landesman, E. & Meyers, D. <u>On saddle point problems in the</u> <u>Calculus of Variations, the Ritz algorithm and monotone convergence.</u> J. Math. Anal. Appl. 52 (1975) 594-614.

[14] Castro, A. <u>Hammerstein integral equations with indefinite kernel.</u> Intern. J. Math. and Math. Sci. 1 (1978) 182-201.

[15] Clark, D. <u>A variant of the Ljusternik-Schnirelman theory.</u> Ind. Univ. Math. J. 2 (1972) 65-74.

JVAG/jik

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2515	2. GOVT ACCESSION NO. AD·A130492	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A MULTIPLICITY RESULT FOR A SEMILINEAR DIRICHLET PROBLEM		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. V. A. Goncalves		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE April 1983
		13. NUMBER OF PAGES 15
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Semilinear Dirichlet Problem, Variational Methods, Maximum Principles.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let Ω be a bounded smooth domain in $\mathbf{R}^N$. We give an estimate for the number of solutions of the problem

$$Lu + g(u) = \lambda u \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

where L is a second order elliptic operator. The behavior of the nonlinearity g both at 0 and at ∞ and the relationship between λ and the spectrum of L play an important role in the analysis.

END

FILMED

8-83

DTIC